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FASTER CALCULATION METHOD FOR UNSTEADY FLOW IN TUBE

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Abstract: Several methods are known for the calculation of unsteady flow in long tubes having a small-diameter. In case of long pipes having a small-diameter radial change of status indicators are neglected, we consider only the tube longitudinal changes. Most of the calculation methods are based on the finite difference method or the method of equal scale interval characteristic. The common feature of these methods is that the condition for their stability is the fulfilment of the Courant-Friedrich-Lewy condition. This paper shows a faster method for calculation unsteady flow in tube. The governing equations are reduced to three first-order quasi-linear ordinary differential equations. They are solved on the time scale interval analytically. The quickness of this method is given by the used stability condition.
Keywords: unsteady flow, CFD, stability condition

INTRODUCTION

Several methods are known for the calculation of unsteady flow in long tubes having a small-diameter. In case of long pipes having a small-diameter radial change of status indicators are neglected, we consider only the tube longitudinal changes [1]. Most of the calculation methods are based on the finite difference method or the method of equal scale interval characteristic. The common feature of these methods is that the condition for their stability is the fulfilment of the Courant-Friedrich-Lewy condition. This means that for a given spacing step the time step has to fulfil the following equation:

$$\Delta t \le \frac{\Delta x}{\max(a+w)}, \qquad (1)$$

i.e. the time scales must be less than or equal to the spacing scale divided by the maximum of the sum of the speed of sound and speed of flow. In this paper we show the correlations for frictionless flow in horizontal tube.

THE GOVERNING EQUATIONS

The continuity equation:

$$\frac{d\rho}{dt} + \rho\frac{\partial w}{\partial x} = 0. \qquad (2)$$

The equation of motion:

$$\frac{dw}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \qquad (3)$$

Energy equation (Thermodynamics I.):

$$\frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \frac{4k}{D\rho}(T_k - T). \qquad (4)$$

Thermodynamic properties of fluid:

$$p = p(\rho, T), \qquad (5)$$

$$h = h(p, \rho). \qquad (6)$$

Based on (6)

$$dh = \left.\frac{\partial h}{\partial p}\right|_\rho dp + \left.\frac{\partial h}{\partial \rho}\right|_p d\rho. \qquad (7)$$

can be written.

Using (7) in the equation (4) it can be written as follows:

$$\left(\left.\frac{\partial h}{\partial p}\right|_\rho - \frac{1}{\rho}\right)\frac{dp}{dt} + \left.\frac{\partial h}{\partial \rho}\right|_p \frac{d\rho}{dt} = \frac{4k}{D\rho}(T_k - T), \qquad (8)$$

after rearranging it we get

$$\frac{\left(\left.\frac{\partial h}{\partial p}\right|_\rho - \frac{1}{\rho}\right)}{\left.\frac{\partial h}{\partial \rho}\right|_p}\frac{dp}{dt} + \frac{d\rho}{dt} = \frac{4k}{D\rho\left.\frac{\partial h}{\partial \rho}\right|_p}(T_k - T). \qquad (9)$$

Using the

$$\frac{\left(\left.\frac{\partial h}{\partial p}\right|_\rho - \frac{1}{\rho}\right)}{\left.\frac{\partial h}{\partial \rho}\right|_p} = -\frac{1}{\left.\frac{\partial p}{\partial \rho}\right|_s} = -\frac{1}{a^2} \qquad (10)$$

correlation

$$a^2\frac{d\rho}{dt} - \frac{dp}{dt} = \frac{4a^2k}{D\rho\left.\frac{\partial h}{\partial \rho}\right|_p}(T_k - T) = b_3. \qquad (11)$$

can be written

Expressed in equation (2)

$$\frac{d\rho}{dt} = -\rho\frac{\partial w}{\partial x}, \qquad (12)$$

and substituting it into equation (11) we get

$$-a^2\rho\frac{\partial w}{\partial x}-\frac{dp}{dt}=b_3, \qquad (13)$$

i.e.

$$-a^2\rho\frac{\partial w}{\partial x}-\frac{\partial p}{\partial t}-w\frac{\partial p}{\partial x}=b_3. \qquad (14)$$

Taking the equation (14) and adding it we get $a\rho$ -times the equation (3), i.e.

$$\left.\begin{aligned}-a^2\rho\frac{\partial w}{\partial x}-\frac{\partial p}{\partial t}-w\frac{\partial p}{\partial x}=b_3\\ \frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad /\cdot a\rho\end{aligned}\right\}+, \qquad (15)$$

to give the

$$a\rho\left[\frac{\partial w}{\partial t}+(w-a)\frac{\partial w}{\partial x}\right]-\left[\frac{\partial p}{\partial t}+(w-a)\frac{\partial p}{\partial x}\right]=b_3 \qquad (16)$$

correlation. This means that along the characteristic (line)

$$\frac{dx}{dt}=w-a \qquad (17)$$

the following ordinary differential equation is satisfied:

$$a\rho\frac{dw}{dt}-\frac{dp}{dt}=b_3. \qquad (18)$$

Similarly, let's consider now the equation (14) and subtract the $a\rho$ -times the equation (3) from it:

$$\left.\begin{aligned}-a^2\rho\frac{\partial w}{\partial x}-\frac{\partial p}{\partial t}-w\frac{\partial p}{\partial x}=b_3\\ \frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad /\cdot a\rho\end{aligned}\right\}- \qquad (19)$$

and multiplying the correlation obtained by (-1):

$$a\rho\left[\frac{\partial w}{\partial t}+(w+a)\frac{\partial w}{\partial x}\right]+\left[\frac{\partial p}{\partial t}+(w+a)\frac{\partial p}{\partial x}\right]=-b_3 \qquad (20)$$

This means that along the characteristic (line)

$$\frac{dx}{dt}=w+a \qquad (21)$$

the following ordinary differential equation is satisfied:

$$a\rho\frac{dw}{dt}+\frac{dp}{dt}=-b_3. \qquad (22)$$

The equation system consisting of the partial differential equations (2), (3) and (11) is the following:

$$\left.\begin{aligned}\frac{\partial\rho}{\partial t}+w\frac{\partial\rho}{\partial x}+\rho\frac{\partial w}{\partial x}=0\\ \frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0\\ a^2\left(\frac{\partial\rho}{\partial t}+w\frac{\partial\rho}{\partial x}\right)-\frac{\partial p}{\partial t}-w\frac{\partial p}{\partial x}=b_3\end{aligned}\right\} \qquad (23)$$

and we get its solution from solving the (11), (18) and (22) ordinary differential equations along the corresponding characteristics:

$$\left.\begin{aligned}a^2\frac{d\rho}{dt}-\frac{dp}{dt}=b_3 \quad \frac{dx}{dt}=w\\ a\rho\frac{dw}{dt}+\frac{dp}{dt}=-b_3 \quad \frac{dx}{dt}=w+a\\ a\rho\frac{dw}{dt}-\frac{dp}{dt}=b_3 \quad \frac{dx}{dt}=w-a\end{aligned}\right\}. \qquad (24)$$

MATHEMATICAL BACKGROUND

Let's consider the following partial differential equation [2], where $u=u(x,t)$ and where c and k are constants:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=k, \qquad (25)$$

Let's take $c>0$, and make the following initial and boundary conditions known.

Initial condition is the following if $x\geq 0$:

$$u(x,0)=f(x), \qquad (26)$$

and boundary condition is at $x=0$:

$$u(0,t)=g(t). \qquad (27)$$

Let's formulate the total differential of function u:

$$du=\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx \qquad (28)$$

and when expressed we get the total derivative of u by t:

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{dx}{dt}\frac{\partial u}{\partial x}. \qquad (29)$$

Comparing the left-hand side of equation (25) and the right side of equation (29) we can write

$$\frac{du}{dt}=k \qquad (30)$$

and

$$\frac{dx}{dt}=c. \qquad (31)$$

Solving the ordinary differential equations (30)&(31), the solution is

$$u=kt+F(x) \qquad (32)$$

and

$$x=ct+x_0, \qquad (33)$$

where x_0 is the location coordinate in the $t=0$ moment.

Based on initial condition (26) the value of u in the $t=0$ moment is:

$$u(x,0)=F(x_0)=f(x_0). \qquad (34)$$

So the solution of the initial value problem is

$$u(x,t)=kt+f(x_0), \qquad (35)$$

when $x_0\geq 0$.

Denoted x_0 from (33) formula

$$u(x,t)=kt+f(x-ct), \qquad (36)$$

can be written when $x-ct\geq 0$.

If $x_0=x-ct<0$, then the solution is calculated from the boundary condition according to following (Figure 1) [3]:

$$u(x,t)=k\frac{x}{c}+g\left(t-\frac{x}{c}\right). \qquad (37)$$

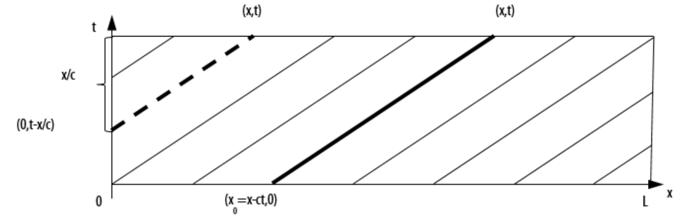

***Figure 1**. Characteristics*

If $c<0$ *, then at location* $x=L$ *we specify the boundary condition, i.e.*

$$u(L,t)=g(t). \tag{38}$$

In this case the solution is the following:

$$u(x,t)=kt+f(x_0)=kt+f(x-ct), \tag{39}$$

when $x_0=x-ct\le L$ *, and*

$$u(x,t)=k\frac{x-L}{c}+g\left(t-\frac{x-L}{c}\right), \tag{40}$$

when $x_0=x-ct>L$ *.*

If the c is constant, the intersection of characteristics is not possible. If the $c=c(x,t)$ *is a function, then differential equation (31) has only one solution for the given* $[0,t]$ *time interval (i.e. the characteristics do not intersect each other [4]), only if the function* $c=c(x,t)$ *can fulfil the Lipschitz condition. The Lipschitz condition is as follows [5]:*

$$|c(x_2,t)-c(x_1,t)|\le L_C|x_2-x_1|, \tag{41}$$

where $L_C>0$ *. Using that,* $dx=x_2-x_1$ *, and*

$c(x_2,t)=c(x_1,t)+\frac{\partial c}{\partial x}dx$ *we get*

$$\left|\frac{\partial c}{\partial x}\right|\le L_C. \tag{42}$$

Expressing c from equation (33) we get

$$c=\frac{x-x_0}{t}. \tag{43}$$

Using this the Lipschitz condition reformulates as follows:

$$\left|\frac{x_2-x_0}{t}-\frac{x_1-x_0}{t}\right|=\left|\frac{x_2-x_1}{t}\right|\le L_C|x_2-x_1| \tag{44}$$

and rearranging it we get

$$L_C\ge\left|\frac{1}{t}\right|. \tag{45}$$

Comparing the equations (42) and (45) it can be written that the function c fulfils the Lipschitz condition when

$$\left|\frac{\partial c}{\partial x}\right|t<1 \tag{46}$$

correlation is met.

THE SOLUTION OF THE SYSTEM OF EQUATIONS

Returning to system of equations (24), let's consider the equations quasi-linear and quasi-constant coefficient equations. Quasi-linear and quasi-constant coefficient equations mean the coefficients are functions but now their values are constant as at the beginning of time interval. Let's solve the system of equations in the time interval $t=0$ *and* $t=\Delta t$ *and on the location interval* $x=0$ *and* $x=L$ *.*

Introducing the following notation:

$$a_1^2\rho-p=u_1, \tag{47}$$

$$w_1=c_1, \tag{48}$$

$$b_3=k, \tag{49}$$

$$a_2\rho_2w+p=u_2, \tag{50}$$

$$w_2+a_2=c_2, \tag{51}$$

$$a_3\rho_3w-p=u_3, \tag{52}$$

$$w_3-a_3=c_3. \tag{53}$$

Based on them system of equations (24) can be written as follows:

$$\frac{\partial u_1}{\partial t}+c_1\frac{\partial u_1}{\partial x}=k, \tag{54}$$

$$\frac{\partial u_2}{\partial t}+c_2\frac{\partial u_2}{\partial x}=-k, \tag{55}$$

$$\frac{\partial u_3}{\partial t}+c_3\frac{\partial u_3}{\partial x}=k. \tag{56}$$

Thus the system is falling into three partial differential equations and according to above their solutions are the followings:

$$u_1(x,t)=\begin{cases}kt+f_1(x_0)\,\text{ha}\;c_1t\le x\le L+c_1t\\ k\frac{x}{c_1}+g_{1,0}\left(t-\frac{x}{c_1}\right)\text{ha}\;x<c_1t\\ k\frac{x-L}{c_1}+g_{1,L}\left(t-\frac{x-L}{c_1}\right)\text{ha}\;x>L+c_1t\end{cases} \tag{57}$$

$$u_2(x,t)=\begin{cases}-kt+f_2(x_0)\,\text{ha}\;x\ge c_2t\\ -k\frac{x}{c_2}+g_{2,0}\left(t-\frac{x}{c_2}\right)\text{ha}\;x<c_2t\end{cases} \tag{58}$$

$$u_3(x,t)=\begin{cases}kt+f_3(x_0)\,\text{ha}\;x\le L+c_3t\\ k\frac{x-L}{c_3}+g_{3,L}\left(t-\frac{x-L}{c_3}\right)\text{ha}\;x>L+c_3t\end{cases} \tag{59}$$

Adding together the equations (50) and (52) and arranging it we get

$$w=\frac{u_2+u_3}{a_2\rho_2+a_3\rho_3}. \tag{60}$$

Subtracting equation (52) from equation (50), arranging it and using relation (60) we get

$$p=\frac{a_3\rho_3u_2-a_2\rho_2u_3}{a_2\rho_2+a_3\rho_3}. \tag{61}$$

In terms of the equation (47) and by using relation (61) we get the following formula for density:

$$\rho=\frac{1}{a_1^2}\left(u_1+\frac{a_3\rho_3u_2-a_2\rho_2u_3}{a_2\rho_2+a_3\rho_3}\right)=\frac{a_3\rho_3(u_1+u_2)+a_2\rho_2(u_1-u_3)}{a_1^2(a_2\rho_2+a_3\rho_3)} \tag{62}$$

The uniqueness of the solution is ensured by the fulfilment of the Lipschitz condition. In this case equation (46) is the following:

$$\max\left(\left|\frac{\partial c_1}{\partial x}\right|,\left|\frac{\partial c_2}{\partial x}\right|,\left|\frac{\partial c_3}{\partial x}\right|\right)\Delta t<1. \tag{63}$$

Switching over from differentials to differences can be written that

$$\Delta t < \frac{\Delta x}{\max(|\Delta c_1|,|\Delta c_2|,|\Delta c_3|)} \qquad (64)$$

must be met.

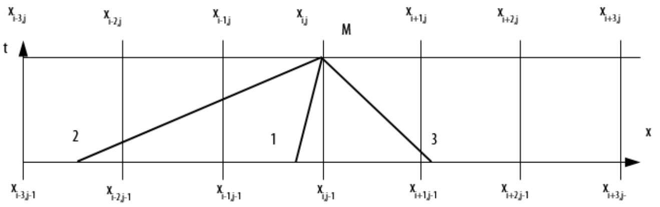

***Figure 2**. Symbols of characteristics*

BOUNDARY CONDITIONS

If point M is at the inlet of pipe and inflow is here, only the characteristic line from point 3 exists (Figure 3), and according to these and based on the equation (59) the relation between speed and pressure at point M must be able to meet the following,:

$$p = p_3 + a_3\rho_3(w - w_3) - k\Delta t\,. \qquad (65)$$

This means that if the speed and density are given, then the pressure can be calculated or if the pressure and density are given, the speed can be calculated.

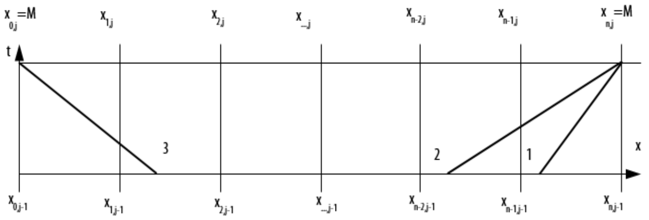

***Figure 3.** Boundary conditions*

If point M is at the end of the pipe and outflow is here, only the characteristics that depart from points 1 and 2 exist. If here the speed is given, then according to (58) the pressure- and according to (57) the density can be calculated, namely as follows:

$$p = p_2 + a_2\rho_2(w_2 - w) - k\Delta t\,, \qquad (66)$$

$$\rho = \rho_1 + \frac{1}{a_1^2}(p - p_1) + k\Delta t\,. \qquad (67)$$

It is taken as a special case when inflow is not at the inlet of pipe. It means that $w_{0,j} = 0$ (Figure 4.).

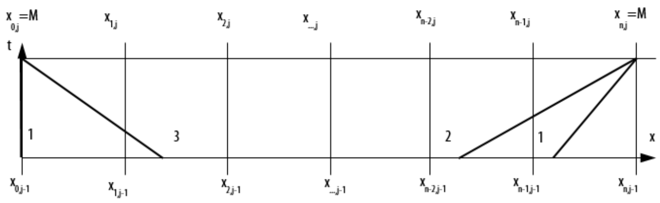

***Figure 4**. Boundary conditions without inflow*

In this case the characteristic that depart from point 1 also exists at the inlet of pipe. Thus, the properties of point M that is at inlet of pipe can be computed as follows:

$$w = 0\,, \qquad (68)$$

$$p = p_3 - a_3\rho_3 w_3 - k\Delta t\,, \qquad (69)$$

$$\rho = \rho_1 + \frac{k\Delta t + (p - p_1)}{a_1^2}\,. \qquad (70)$$

CONCLUSIONS

The essence of the method presented here is that the system of equations which describes the flow is reduced to three first-order quasi-linear partial differential equations, which are solved on the Δt time interval where the coefficients of equations are calculated from the status indicators that are known at the beginning of the time interval. The quickness of this method is given by the used stability condition. Here Lipschitz condition (64) must be used instead of Courant-Friedrichs-Lewy condition (1). This means that the calculated time scale for the fixed space scale is not related to the absolute value of the speed of sound and that of the flow speed only their rate of change. In a particular case $(a_{max} = 391{,}8\,m/s;\, w_{max} = 40{,}3\,m/s)$ of Figure 5 shows the time function of the number of calculation step.

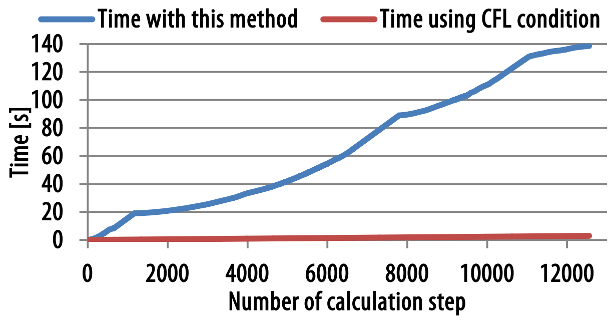

***Figure 5**. Comparison of methods*

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