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STUDY OF DIFFERENTIAL EQUATIONS WITH THEIR POLYNOMIAL AND NONPOLYNOMIAL SPLINE BASED APPROXIMATION

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Abstract: *The purpose of this paper is to discuss numerical solutions of differential equations including the evolution, progress and types of differential equations. Special attention is given to the solution of differential equations by application of spline functions. Here we are interested in differential equation based problems and their solutions using polynomial and nonpolynomial splines of different orders. It contains crux of various recent research papers based on application of splines of different orders.*
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INTRODUCTION

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions – the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. In general, it is not possible to obtain the analytical solution of a system of differential equations, obtained from obstacle, unilateral, moving and free boundary-value problems and problems of the defection of plates and in a number of other scientific applications, while many numerical methods have been developed to determine solutions with a given degree of accuracy. In the present paper we discuss the history, classification and numerical solution of differential equations. Here we are merely concerned about the solution of these boundary-value problems by application of spline functions.

We include here one paper based on application of various spline functions to solve different systems of differential equations. The paper is organized as follows: in section 2, we will discuss a brief history of differential equations. In section 3, we consider the general introduction of differential equations. In section 4, we discuss about types of differential equations. In section 5, subdivision of differential equations, in section 6, initial & boundary value problems, in section 7 types of boundary value problems, in section 8 differential equations associated with physical problems arising in engineering is discussed. In section 9, numerical solution of differential equations, in section 10, general introduction to spline, in section 11 spline solution of differential equations and finally in section 12 the conclusion and further development is given.

HISTORY OF DIFFERENTIAL EQUATIONS

The study of differential equations is a wide field. Many of the laws in physics, chemistry, engineering, biology and economics are based on empirical observations that describe changes in the states of systems. Mathematical models that describe the state of such systems are often expressed in terms of not only certain system parameters but also their derivatives. Such mathematical models, which use differential calculus to express relationship between variables, are known as differential equations.

The history of differential equations traces the development of differential equations from calculus, which was independently invented by

English physicist Isaac Newton (1665) and German mathematician Gottfried Leibnitz (1674). The term differential equation was coined by Leibnitz in 1676 for a relationship between the two differentials dx and dy for the two variables x and y. Newton solved his first differential equation in 1676 by the use of infinite series, eleven years after his discovery of calculus in 1665. Leibnitz solved his first differential equation in 1693, the year in which Newton first published his results. Hence, 1693 marks the inception for the differential equations as a distinct field in mathematics [1].

The different phases of 17^{th}, 18^{th} and 19^{th} Centuries played some crucial role in the history of differential equations. In the year 1695 the problem of finding the general solution of what is now called Bernoulli's equation was proposed by Bernoulli and it was solved by Leibnitz and Johann Bernoulli by different methods. In further development 1724 was important to the early history of ordinary differential equations. Ordinary differential equation acquired its significance when it was introduced in 1724 by Jacopo Francesco, Count Ricatti of Venice in his work in acoustics. Further in the year of 1739 Leonhard Euler solves the general homogeneous linear ordinary differential equation with constant coefficients. L'Hospital came up with separation of variables in 1750, and it is now the physicist's handiest tool for solving partial differential equations. Since its introduction in 1828, Green's functions have become a fundamental mathematical technique for solving boundary-value problems. In 1890 Poincare [2] gave the first complete proof of the existence and uniqueness of a solution of the Laplace Equation for any continuous Dirichlet boundary condition.

In 20^{th} Century a lot of quality work has been done in the field of differential equations, but the major concern was the analytic and computational solution of differential equations. In last few decades numerical analysis of differential equations has become a major topic of study. In view of this, this thesis gives a small step towards the development of computational analysis of ordinary differential equations, which have lot of utilities in the field of science and engineering.

GENERAL INTRODUCTION OF DIFFERENTIAL EQUATIONS

There is difference between differential equations and ordinary equations of mathematics. The differential equations, in addition to variables and constants, also contain derivatives of one or more of the variables involved. In general, a differential equation is an equation which involves the derivatives of an unknown function represented by a dependent variable. It expresses the relationship involving the rates of change of continuously changing quantities modeled by functions and are used whenever a rate of change (derivative) is known. A solution to a differential equation is a function whose derivatives satisfy the equation.

The order and degree are two major terms if we discuss about a differential equation. The order of a differential equation is that of the highest derivative that it contains. For instance, a second-order differential equation contains only second and first derivatives. If a differential equation can be rationalized and cleared of fraction with regard to all derivatives present, the exponent of the highest order derivative is called the degree of the differential equation.

TYPES OF DIFFERENTIAL EQUATIONS

The Differential equations can be categorized in ordinary differential equations (ODE), partial differential equation (PDE), delay differential equation (DDE), stochastic differential equation (SDE) and differential algebraic equation (DAE) which are defined as follows:

- *(a) An ordinary differential equation (ODE) is a differential equation in which the unknown function is a function of a single independent variable.*
- *(b) A partial differential equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and their partial derivatives.*
- *(c) A delay differential equation (DDE) is a differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.*
- *(d) A stochastic differential equation (SDE) is a differential equation in which one or more of the terms are a stochastic process, thus*

resulting in a solution which is itself a stochastic process.

(e) A differential algebraic equation (DAE) is a differential equation comprising differential and algebraic terms, given in implicit form.

An ordinary differential equation (ODE) implicitly describes a function depending on a single variable and the ODE expresses a relation between the solution and one or more of its derivatives. Beside the ODE, usually one or more additional (initial) conditions are needed to determine the unknown function uniquely.

A Partial differential equation (PDE) is a relation involving an unknown function of at least two independent variables and its partial derivatives with respect to those variables. Partial differential equations are used to formulate and solve problems that involve unknown functions of several variables, such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, elasticity or more generally any process that is distributed in space or distributed in space and time. In general, A partial differential equation (PDE) is an equation involving functions and their partial derivatives.

SUBDIVISION OF DIFFERENTIAL EQUATIONS

Each of types of differential equations mentioned above is divided into two subcategories - linear and nonlinear. A differential equation is linear if it involves the unknown function and its derivatives only to the first power; otherwise the differential equation is nonlinear. Thus if y' denotes the first derivative of y, then the equation $y'=y$ is linear, while the equation $y'=y^2$ is nonlinear. Solutions of a linear equation in which the unknown function or its derivative or derivatives appear in each term (linear homogeneous equations) may be added together or multiplied by an arbitrary constant in order to obtain additional solutions of that equation, but there is no general way to obtain families of solutions of nonlinear equations, except when they exhibit symmetries. Linear equations frequently appear as approximations to nonlinear equations, and these approximations are only valid under restricted conditions.

INITIAL & BOUNDARY VALUE PROBLEMS

Ordinary differential equations (ODEs) describe phenomena that change continuously. They arise in models throughout mathematics, science, and engineering. By itself, a system of ODEs has many solutions. Commonly a solution of interest is determined by specifying the values of all its components at a single point $x=a$. This is an initial value problem (IVP). However, in many applications a solution is determined in a more complicated way. A boundary value problem (BVP) specifies values or equations for solution components at more than one x. Unlike IVPs, a boundary value problem may not have a solution, or may have a finite number, or may have infinitely many solutions.

Initial value problem has all of the conditions specified at the same value of the independent variable in the equation (and that value is at the lower boundary of the domain, thus the term "initial" value). On the other hand, a boundary value problem has conditions specified at the extremes of the independent variable. For example, if the independent variable is time over the domain [0,1], an initial value problem would specify a value of $y(t)$ at time 0, while a boundary value problem would specify values for $y(t)$ at both $t = 0$ and $t = 1$.

If the problem is dependent on both space and time, then instead of specifying the value of the problem at a given point for all time the data could be given at a given time for all space. For example, the temperature of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem. Whereas in the middle of a still pond if somebody taps the water with a known force that would create a ripple and give us an initial condition.

TYPES OF BOUNDARY VALUE PROBLEMS

(i) Dirichlet Boundary Condition:

If the boundary gives a value to the problem then it is a Dirichlet boundary condition. For example if one end of an iron rod held at absolute zero then the value of the problem would be known at that point in space. A Dirichlet boundary condition imposed on an ordinary differential equation or a partial differential equation specifies the values a solution is to take on the boundary of the domain. The question of finding solutions to such equations is known as the Dirichlet problem.

For example, in the case of an ordinary differential equation such as

$$\frac{d^2y}{dx^2}+5y=1 \quad (1)$$

on the interval [0,1] the Dirichlet boundary conditions take the form

$$y(0)=\alpha_1 \text{ and } y(1)=\alpha_2 \quad (2)$$

where α_1 and α_2 are given numbers.

(ii) Neumann Boundary Condition:

If the boundary gives a value to the normal derivative of the problem then it is a Neumann boundary condition. For example if one end of an iron rod had a heater at one end then energy would be added at a constant rate but the actual temperature would not be known. A Neumann boundary condition imposed on an ordinary differential equation or a partial differential equation specifies the values the derivative of a solution is to take on the boundary of the domain. In the case of ordinary differential equation such as

$$\frac{d^2y}{dx^2}+5y=1$$

on the interval [0, 1] the Neumann boundary conditions take the form

$$y'(0)=\alpha_1 \text{ and } y'(1)=\alpha_2 \quad (3)$$

where α_1 and α_2 are given numbers.

(iii) Cauchy Boundary Condition:

If the boundary has the form of a curve or surface that gives a value to the normal derivative and the problem itself then it is a Cauchy boundary condition. A Cauchy boundary condition imposed on an ordinary differential equation or a partial differential equation specifies both the values a solution of a differential equation is to take on the boundary of the domain and the normal derivative at the boundary. It corresponds to imposing both a Dirichlet and a Neumann boundary condition.

$$\frac{d^2y}{dx^2}+5y=1$$

Cauchy boundary conditions can be understood from the theory of second order, ordinary differential equations, where to have a particular solution one has to specify the value of the function and the value of the derivative at a given initial or boundary point, i.e.,

$$y(a)=\alpha_1 \text{ and } y'(a)=\alpha_2 \quad (4)$$

where α_1 and α_2 are given numbers and a is a boundary or initial point.

DIFFERENTIAL EQUATIONS ASSOCIATED WITH PHYSICAL PROBLEMS ARISING IN ENGINEERING

As the world turns, things change, mountains erode, river beds change, machines break down, the environment becomes more polluted, populations shift, economics fluctuate, technology advances. Hence any quantity expressible mathematically over a long time must change as a function of time. As a function of time, relatively speaking, there are many quantities which change rapidly, such as natural pulsation of a quartz crystal, heart beats, the swing of a pendulum, chemical explosions, etc.

When we get down to the business of quantitative analysis of any system, our experience shows that the rate of change of a physical or biological quantity relative to time has vital information about the system. It is this rate of change which plays a central role in the mathematical formulation of most of the physical and biological models amenable to analysis.

Engineering problems that are time-dependent are often described in terms of differential equations with conditions imposed at single point (initial/final value problems); while engineering problems that are position dependent are often described in terms of differential equations with conditions imposed at more than one point (boundary value problems). Some of the motivational examples encountering in many engineering fields are as follows:

- *(i) Coupled L-R electric circuits*
- *(ii) Coupled systems of springs*
- *(iii) Motion of a particle under a variable force field*
- *(iv) Newton's second law in dynamics (mechanics)*
- *(v) Radioactive decay in nuclear physics*
- *(vi) Newton's law of cooling in thermodynamics.*
- *(vii) The wave equation*
- *(viii) Maxwell's equations in electromagnetism*
- *(ix) The heat equation in thermodynamics*
- *(x) Laplace's equation, which defines harmonic functions*
- *(xi) The beam deflections equation*
- *(xii) The draining and coating flows equation*

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

The study of differential equations is a wide field in both pure and applied mathematics. Pure mathematicians study the types and properties of *differential equations, such as whether or not solutions exist, and should they exist, whether they are unique.*

Applied mathematicians emphasize differential equations from applications, and in addition to existence/uniqueness questions, are also concerned with rigorously justifying methods for approximating solutions. Physicists and engineers are usually more interested in computing approximate solutions to differential equations. These solutions are then used to simulate celestial motions, simulate neurons, design bridges, automobiles, aircraft, sewers, etc. Often, these equations do not have closed form solutions and are solved using numerical methods.

Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere. This extension is often necessary for solutions to exist, and it also results in more physically reasonable properties of solutions, such as shocks in hyperbolic (or wave) equations.

Numerical techniques to solve the boundary value problems include some of the following methods:

- Shooting Methods: *These are initial value problem methods. In this method, we convert the given boundary value problem to an initial value problem by adding sufficient number of conditions at one end and adjust these conditions until the given conditions are satisfied at the other end.*

- Finite Difference Methods:*In finite difference method (FDM), functions are represented by their values at certain grid points and derivatives are approximated through differences in these values. For the finite difference method, the domain under consideration is represented by a finite subset of points. These points are called "nodal points" of the grid. This grid is almost always arranged in (uniform or non-uniform) rectangular manner. The differential equation is replaced by a set of difference equations which are solved by direct or iterative methods.*

- Finite Element Methods: *In finite element method (FEM), functions are represented in terms of basis functions and the ODE is solved in its integral (weak) form. In this method the domain under consideration is partitioned in a finite set of elements. In this the differential equation is discretized by using approximate methods with the piecewise polynomial solution [3].*

- Spline Based Methods: *In spline based methods, the differential equation is discretized by using approximate methods based on spline. The end conditions are derived for the definition of spline. The algorithm developed not only approximates the solutions, but their higher order derivatives as well.*

GENERAL INTRODUCTION TO SPLINE

Usually a spline is a piecewise polynomial function defined in a region D, such that there exists a decomposition of D into sub-regions in each of which the function is a polynomial of some degree m. Also the function, as a rule, is continuous in D, together with its derivatives of order up to (m-1).

Thus, a spline function of order m, $S_\Delta(x)$, interpolating to a function u(x) defined on [a,b] is such that:

(i) In each subinterval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most m.

(ii) The first, second, third... (m-1)th derivatives of $S_\Delta(x)$ are continuous on [a,b].

In its most general form a polynomial spline $S_\Delta(x)$ consists of polynomial pieces in $[x_i, x_{i+1}]$, i=0,1,2,….N-1 where the given N points are called knots. The vector $(x_0,….x_{N-1})$ is called a knot vector for the spline. If the knots are equidistantly distributed in the interval [a,b] we say the spline is uniform, otherwise we say it is non-uniform.

If the polynomial pieces on the subintervals $[x_i, x_{i+1}]$, i=0,1,2,….N-1, all have degree at most n, then the spline is said to be of degree ≤ n (or of order n+1).

Examples: *Suppose the interval [a,b] is [0,3] and the subintervals are [0,1), [1,2), and [2,3]. Suppose the polynomial pieces are to be of degree 2, and the pieces on [0,1) and [1,2) must join in value and first derivative (at x=1) while the pieces on [1,2) and [2,3] join simply in value (at x=2). This would define a type of spline $S_\Delta(x)$ for which*

$$S_\Delta(x) = -1 + 4x - x^2, 0 \le x < 1$$

$$S_\Delta(x) = 2x, 1 \le x < 1$$

$$S_\Delta(x) = 2 - x + x^2, 2 \le x \le 3$$

would be a member of that type, and

$$S_\Delta(x) = -2 - 2x^2, 0 \le x < 1$$

$$S_\Delta(x) = 1 - 6x + x^2, 1 \le x < 2$$

$$S_{\Delta}(x) = -1 + x - 2x^2, 2 \le x \le 3$$

would also be a member of that type.

The simplest spline has degree 0. It is also called a step function. The next simplest spline has degree 1. It is called a linear spline. The spline of degree 2 is called quadratic spline.

The literature of splines is replete with names for special types of splines. These names have been associated with the choices made for representing the spline or the choices made in forming the extended knot vector or any special conditions imposed on the spline or the choice of introducing a parameter such as:

- *(i) B – splines is obtained using basis B-splines as basis functions for the entire spline .*
- *(ii) Bezier splines is obtained using Bernstein polynomials as employed by Pierre Bézier to represent each polynomial piece.*
- *(iii) Uniform splines is obtained using single knots for C^{n-1} continuity and spacing these knots evenly on [a,b].*
- *(iv) Non-uniform splines is obtained using knots with no restriction on spacing.*
- *(v) Natural splines is obtained enforcing zero second derivatives at end values a and b.*
- *(vi) Interpolating splines are requiring that given data values be on the spline.*
- *(vii) Polynomial spline is a piecewise polynomial function defined in a region D, such that there exists a decomposition of D into sub-regions in each of which the function is a polynomial of some degree m.*

The connecting polynomials could be of any degree and therefore we have different types of spline functions such as linear, quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic etc. They are also known as 'polynomial spline' function.

(viii) Nonpolynomial spline

To deal effectively with problems we introduce 'spline functions' containing a parameter τ. These are 'non-polynomial splines'. These 'splines' belong to the class C^2 and reduce into polynomial splines as parameter $\tau \to 0$ [4, 5].

SPLINE SOLUTION OF DIFFERENTIAL EQUATIONS

In the study of problems arising in astrophysics, problem of heating of infinite horizontal layer of fluid, eigenvalue problems arising in thermal instability, obstacle, unilateral, moving and free boundary-value problems, problems of the defection of plates and in a number of other scientific applications, we find a system of differential equations of different order with different boundary conditions. In general, it is not possible to obtain the analytical solution of them; we usually resort to some numerical methods for obtaining an approximate solution of these problems.

In the present paper various spline techniques for solving boundary value problems in ordinary differential equations are briefly discussed. It contains crux of various recent research papers based on application of quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic and other higher order spline functions to solve different systems of ordinary differential equations. The recent spline techniques, which are used frequently in various fields like; biology, physics and engineering, are considered in this paper.

Quadratic Spline Techniques to Solve Boundary Value Problems:

A quadratic spline function $S_{\Delta}(x)$, interpolating to a function u(x) defined on [a,b] is such that

- *(i) In each subinterval $[x_{j-1}, x_j]$=1,2,….,N, $S_{\Delta}(x)$ is a polynomial of degree at most two.*
- *(ii) The first derivative of $S_{\Delta}(x)$ is continuous on [a,b].*

Considering the paper [6] by Siraj-ul-Islam et al. having the system of second-order boundary value problem of the type

$$y'' = \begin{cases} f(x), & a \le x \le c \\ g(x)y(x) + f(x) + r, & c \le x \le d \\ f(x), & d \le x \le b \end{cases} \qquad (5)$$

with the boundary conditions

$$y(a) = \alpha_1 \text{ and } y(b) = \alpha_2 \qquad (6)$$

and assuming the continuity conditions of y and y′ at c and d. Here, f and g are continuous functions on [a,b] and [c,d] respectively. The parameters r, α_1, α_2 are real finite constants.

In this research article, quadratic non-polynomial spline functions are used to develop a numerical method for obtaining smooth approximations to the solution of a system of second-order boundary-value problems of the type (5). The new method is of order two for arbitrary α and β if $2\alpha + 2\beta - 1 = 0$ and method of order 4 if $\alpha = 1/12$ along with $2\alpha + 2\beta - 1 = 0$.

Cubic Spline Techniques to Solve Boundary Value Problems:

A cubic spline function $S_\Delta(x)$ of class $C^2[a,b]$ interpolating to a function $u(x)$ defined on $[a,b]$ is such that

(a) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most three,

(b) The first and second derivatives of $S_\Delta(x)$ are continuous on [a,b].

Considering the article [7] by E. A. Al-Said having the system of second-order boundary value problem of the type

$$u''=\begin{cases} f(x), & a\le x\le c \\ g(x)u(x)+f(x)+r, & c\le x\le d \\ f(x), & d\le x\le b \end{cases} \qquad (7)$$

with the boundary conditions

$$u(a)=\alpha_1 \text{ and } u(b)=\alpha_2 \qquad (8)$$

and assuming the continuity conditions of u and u′ at c and d. Here, f and g are continuous functions on [a,b] and [c,d], respectively. The parameters r, α_1, α_2 are real finite constants.

The main purpose of this article is to use uniform cubic spline functions to develop some consistency relations which are then used to develop a numerical method for computing smooth approximations to the solution and its derivatives for a system of second-order boundary-value problems of the type (7). In this paper the author has shown that the present method gives approximations which are better than those produced by other collocation, finite-difference, and spline methods.

Quartic Spline Techniques to Solve Boundary Value Problems:

A quartic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ defined on [a,b] is such that

(i) In each subinterval $[x_{j-1}, x_j]$=1,2,….,N,, $S_\Delta(x)$ is a polynomial of degree at most four.

(ii) The first, second and third derivatives of $S_\Delta(x)$ are continuous on [a,b].

Considering the paper by E. A. Al-Said [8] having the system of fourth order boundary value problem of the type

$$u^{(iv)}=\begin{cases} f(x), & a\le x\le c \\ g(x)u(x)+f(x)+r, & c\le x\le d \\ f(x), & d\le x\le b \end{cases} \qquad (9)$$

with the boundary conditions

$$\begin{aligned} &u(a)=u(b)=\alpha_1 \text{ and } u''(a)=u''(b)=\alpha_2, \\ &u(c)=u(d)=\beta_1 \text{ and } u''(c)=u''(d)=\beta_2, \end{aligned} \qquad (10)$$

where f and g are continuous functions on [a,b] and [c,d] respectively. The parameters r, αi and β_i, i=1,2 are real constants.

In the this paper, the authors have used the quartic spline functions to develop a new numerical technique for obtaining smooth approximations of the solution of (9) and its first, second and third derivatives. They derived the consistency relations and developed the new quartic spline method. The convergence analysis of the method, the numerical experiments and comparison with other methods are discussed.

Quintic Spline Techniques to Solve Boundary Value Problems:

A quintic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on [a,b] is defined as:

(i) In each subinterval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most five.

(ii) The first, second, third and fourth derivatives of $S_\Delta(x)$ are continuous on [a,b].

To be able to deal effectively with such problems we introduce 'spline functions' containing a parameter ω. These are 'non-polynomial splines' defined through the solution of a differential equation in each subinterval. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. These 'splines' belong to the class $C^4[a,b]$ and reduce into polynomial splines as parameter $\omega\to 0$. A paper based on quintic spline is as follows -

Considering the paper by Arshad Khan and Tariq Aziz [9] having a third-order linear and non-linear boundary value problem of the type

$$y'''(x)=f(x,y),\ \mathrm{a\le x\le b}, \qquad (11)$$

Subject to

$$\mathrm{y(a)}=\mathrm{k}_1,\ y'(a)=k_2,\ y(b)=k_3. \qquad (12)$$

In this paper, the authors have derived a fourth order method to solve third-order linear and non-linear BVPs using quintic splines. They presented the formulation of their method for third-order linear and non-linear BVPs. To retain the pentadiagonal structure of the coefficient matrix, they derived fourth order boundary equations.

In this paper, the methods discussed are tested on two problems from the literature [10], and absolute errors in the analytical solutions are calculated.

The results confirm the theoretical analysis of the methods. For the sake of comparisons, the authors also tabulated the results by the method of Caglar et al. [11]. We have also applied nonpolynomial quintic spline [12-15] to solve second order linear differential equations.

Sextic Spline Techniques to Solve Boundary Value Problems:

A sextic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ is defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most six.

(ii) The first fifth derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

Consider the paper [16] having a system of second-order boundary-value problem of the type (7) and (8). The authors J. Rashidinia et al. have developed a new numerical method for solving a system of second-order boundary-value problems based on sextic spline. They have shown that the results obtained are very encouraging and their method has better numerical results than those produced by collocation, finite difference and splines methods when solving (7).

Here the authors have considered the obstacle boundary-value problem of finding y such that, on $\Omega=[0,\pi]$,

$$\begin{aligned} -y'' &\geq f(x), \\ y(x) &\geq \psi(x), \\ [y''-f(x)][y(x)-\psi(x)] &= 0, \\ y(0) = y(\pi) &= 0, \end{aligned} \tag{13}$$

Where f(x) is a given force acting on the string and Ψ(x) is the elastic obstacle. The authors have studied problem (13) in the framework of a variational inequality approach. It can be shown (see for example [17-21]) that the problem (13) is equivalent to the variational inequality problem:

$$a(y,v-y)\geq(f,v-y), \text{ for all } v\in K,$$

where K is the closed convex set $K=\{v: v\in H_0^1(\Omega), v\geq\psi \text{ on } \Omega\}$. *This equivalence has been used to study the existence of a unique solution of (13).*

Septic Spline Technique to Solve Boundary Value Problems:

A septic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on [a,b] is defined as:

(i) In each interval $[x_{j-1}, x_j]$, $S_\Delta(x)$ is a polynomial of degree at most seven.

(ii) The first six derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

In a nonpolynomial septic spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class $C^6[a,b]$ and reduces into polynomial splines as parameter $k\to 0$.

Considering the paper by Ghazala Akram et al. [22] having the system of sixth-order boundary value problem of the type

$$\left.\begin{aligned} &y^{(6)}(x)+f(x)y(x)=g(x), \qquad x\in[a,b], \\ &y(a)=\alpha_0, y(b)=\alpha_1, \\ &y^{(1)}(a)=\gamma_0, y^{(1)}(b)=\gamma_1, \\ &y^{(2)}(a)=\delta_0, y^{(2)}(b)=\delta_1, \end{aligned}\right\} \tag{14}$$

where α_i,γ_i *and* $\delta_i, i=0,1$ *are finite real constants and the functions f(x) and g(x) are continuous on [a,b].*

In the present paper, the authors have applied non-polynomial spline functions that have a polynomial and trigonometric parts to develop a new numerical method for obtaining smooth approximations to the solution of such system of sixth-order differential equations.

The nonpolynomial spline function, under consideration has the form

$$T_n=\text{span}\{1,x,x^2,x^3,x^4,x^5,\cos(kx),\sin(kx)\},$$

where kis taken to be the frequency of the trigonometric part of the spline function. It is to be noted that k can be real or pure imaginary which is used to raise the accuracy of the method. In this paper using derivative continuities at knots, the consistency relation between the values of spline and its sixth order derivatives at knots is determined. The nonpolynomial spline solution approximating the analytic solution of the BVP (14) is determined, using the consistency relation involving the sixth order derivatives and the values of the spline along with the end conditions. The error bound of the solution is also determined.

The method presented in this paper has also been proved to be second order convergent. Two examples are considered for the numerical illustrations of the method developed. The method is also compared with those developed by El-Gamel et al. [23] and Siddiqi and Twizell [24] as well and is observed to be better.

Octic Spline Technique to Solve Boundary Value Problems:

An octic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on [a,b] defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most eight.

(ii) The first seventh derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

Considering the paper by S. S. Siddiqi et al. [25] having the system of eighth-order boundary value problem of the type

$$\begin{aligned} &y^{(\mathrm{viii})}+\phi(x)y+\psi(x) \qquad -\infty<a\le x\le b<\infty,\\ &y(a)=A_0,\ y^{(\mathrm{ii})}(a)=A_2,\ y^{(\mathrm{iv})}(a)=A_4,\ y^{(\mathrm{vi})}(a)=A_6,\\ &y(b)=B_0,\ y^{(\mathrm{ii})}(b)=B_2,\ y^{(\mathrm{iv})}(b)=B_4,\ y^{(\mathrm{vi})}(b)=B_6, \end{aligned} \quad (15)$$

where y=y(x) and Φ(x) and φ(x) are continuous function defined in the interval $x\in$ [a,b]. A_i and Bi, i=0,2,4,6 are finite real constants.

In this paper, the authors have used octic spline to solve the problem of the type (15). The spline function values at the midknots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions, and their higher-order derivatives, of differential equations. Four numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.

Nonic Spline Technique to Solve Boundary Value Problems:

A nonic spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on [a,b] defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most nine.

(ii) The first eighth derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

Considering the paper [26] having the system of eighth-order boundary value problem of the type

$$\left.\begin{aligned} &y^{(8)}(x)+f(x)y(x)=g(x), \qquad x\in[a,b],\\ &y(a)=\alpha_0,\ y(b)=\alpha_1,\\ &y^{(1)}(a)=\gamma_0,\ y^{(1)}(b)=\gamma_1,\\ &y^{(2)}(a)=\delta_0,\ y^{(2)}(b)=\delta_1,\\ &y^{(3)}(a)=\upsilon_0,\ y^{(2)}(b)=\upsilon_1, \end{aligned}\right\} \quad (16)$$

where $\alpha_i,\gamma_i,\delta_i$ and $\upsilon_i, i=0,1$ are finite real constants and the functions f(x) and g(x) are continuous on [a,b].

In the present paper, Ghazala Akram et al. have used Nonic spline for the numerical solutions of the eighth order linear special case boundary value problem given by equation (16). The end conditions are derived for the definition of spline. The algorithm developed not only approximates the solutions, but their higher order derivatives as well. The method presented in this paper has also been proved to be second order convergent. Two examples compared with those considered by Siddiqi et al. [25] and Inc et al. [27], show that the method developed in this paper is more efficient. Collocation method is developed for the approximate solution of eighth order linear special case BVP, using nonic spline. The method is also proved to be second order convergent.

Tenth Degree Spline Technique to Solve Boundary Value Problems:

A Tenth degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on [a,b] defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most ten.

(ii) The first ninth derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

Considering the paper by S. S. Siddiqi et al. [28] having the system of tenth-order boundary value problem of the type

$$\left.\begin{aligned} &y^{(x)}(x)+\phi(x)y=\psi(x), \qquad -\infty<a\le x\le b<\infty,\\ &y(a)=A_0,\ y^{(ii)}(a)=A_2,\\ &y^{(iv)}(a)=A_4,\ y^{(vi)}(a)=A_6,\\ &y^{(viii)}(a)=A_8,\ y(b)=B_0,\ y^{(ii)}(b)=B_2,\\ &y^{(iv)}(b)=B_4,\ y^{(vi)}(b)=B_6,\\ &y^{(viii)}(b)=B_8, \end{aligned}\right\} \quad (17)$$

where y=y(x), Φ(x) and Ψ(x) are continuous function defined in the interval $x\in$ [a,b] and A_i and Bi, i=0,2,4,6,8 are finite real constants.

In the present paper, linear, tenth-order boundary-value problems (special case) are solved, using polynomial splines of degree ten. The spline function values at midknots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions and their higher-order derivatives, of *differential equations. Four numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.*

Eleventh Degree Spline Technique to Solve Boundary Value Problems:

An Eleventh degree spline function $S_\Delta(x)$, interpolating to a function $u(x)$ on $[a,b]$ defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most eleven.

(ii) The first tenth derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

In a nonpolynomial Eleventh degree spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class $C^{10}[a,b]$ and reduces into polynomial splines as $k\to 0$.

Considering the paper [29] having the system of tenth-order boundary value problem of the type

$$\left.\begin{aligned} &y^{(10)}(x)+f(x)y(x)=g(x), \qquad x\in[a,b],\\ &y(a)=\alpha_0,\, y(b)=\alpha_1,\\ &y^{(1)}(a)=\gamma_0,\, y^{(1)}(b)=\gamma_1,\\ &y^{(2)}(a)=\delta_0,\, y^{(2)}(b)=\delta_1,\\ &y^{(3)}(a)=\upsilon_0,\, y^{(2)}(b)=\upsilon_1,\\ &y^{(4)}(a)=\zeta_0,\, y^{(4)}(b)=\zeta_1, \end{aligned}\right\} \qquad (18)$$

where $\alpha_i,\gamma_i,\delta_i,\upsilon_i$ and $\zeta_i, i=0,1$ are finite real constants and the functions f(x) and g(x) are continuous on [a,b].

In this paper, S. S. Siddiqi et al. have obtained numerical solutions of the tenth-order linear special case boundary value problems using eleventh degree spline. The end conditions consistent with the BVP are also derived. Siddiqi and Twizell [28] presented the solutions of tenth-order boundary value problems using tenth degree spline, where some unexpected results for the solution and higher order derivatives were obtained near the boundaries of the interval. No such unexpected situation is observed in this method, near the boundaries of the interval and the results are better in the whole interval. The algorithm developed approximates the solutions, and their higher order derivatives. Numerical illustrations are tabulated to compare the errors with those considered by Siddiqi and Twizell [27] and the method is observed to be better.

Twelfth Degree Spline Technique to Solve Boundary Value Problems:

A twelfth degree spline function $S_\Delta(x)$, interpolating to a function u(x) on [a,b] defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most twelve.

(ii) The first eleven derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x_i)=u(x_i)$, i=0(1)N+1.

Considering the paper [30] having the system of twelfth-order boundary value problem of the type

$$\begin{aligned} &y^{(xii)}+\phi(x)y=\psi(x), \qquad -\infty<a\le x\le b<\infty\\ &y^{(2k)}(a)=A_{2k},\, y^{(2k)}(b)=B_{2k}, \qquad k=0,1,2,....,5 \end{aligned} \qquad (19)$$

where y=y(x), Φ(x) and Ψ(x) are continuous function defined in the interval $x\in[a,b]$ and A_i and Bi, i=0,2,4,6,8,10 are finite real constants.

In the present paper, S. S. Siddiqi et al. have solved linear twelfth-order boundary-value problems (special case), using polynomial splines of degree twelve. The spline function values at midknots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations. The algorithm developed approximates the solutions and their higher-order derivatives, of differential equations. Two numerical illustrations are given to show the practical usefulness of the algorithm developed. It is observed that this algorithm is second-order convergent.

Thirteenth Degree Spline Technique to Solve Boundary Value Problems:

A thirteen degree spline function $S_\Delta(x)$, interpolating to a function u(x) on [a,b] defined as:

(i) In each interval $[x_{j-1}, x_j]$=1,2,….,N, $S_\Delta(x)$ is a polynomial of degree at most thirteen.

(ii) The first twelve derivatives of $S_\Delta(x)$ are continuous on [a,b].

(iii) $S_\Delta(x)=u(x)$, i=0(1)N+1.

In a nonpolynomial thirteenth degree spline we introduce a parameter k. The arbitrary constants are being chosen to satisfy certain smoothness conditions at the joints. This 'spline' belongs to the class $C^{12}[a,b]$ and reduces into polynomial splines as $k\to 0$.

Considering the paper by S. S. Siddiqi et al. [31] having the system of twelfth-order boundary value problem of the type

$$\left.\begin{aligned} & y^{(12)}(x)+f(x)y(x)=g(x), \qquad x\in[a,b],\\ & y(a)=\alpha_0,\, y(b)=\alpha_1,\\ & y^{(1)}(a)=\gamma_0,\, y^{(1)}(b)=\gamma_1,\\ & y^{(2)}(a)=\delta_0,\, y^{(2)}(b)=\delta_1,\\ & y^{(3)}(a)=\upsilon_0,\, y^{(2)}(b)=\upsilon_1,\\ & y^{(4)}(a)=\zeta_0,\, y^{(4)}(b)=\zeta_1,\\ & y^{(5)}(a)=\omega_0,\, y^{(5)}(b)=\omega_1, \end{aligned}\right\} \quad (20)$$

where $\alpha_i, \gamma_i, \delta_i, \upsilon_i, \zeta_i$ and $\omega_i, i = 0,1$ are finite real constants and the functions f(x) and g(x) are continuous on [a,b].

In this paper numerical solutions of the twelfth order linear special case boundary value problems are obtained using thirteen degree spline. The end conditions are derived for the definition of spline. Siddiqi and Twizell [30] presented the solutions of twelfth order boundary value problems using 12th degree spline, where some unexpected results for the solution and its derivatives were obtained, near the boundaries of the interval. No such situation is observed in this method. The algorithm developed, approximates not only the solution but its higher order derivatives as well. Numerical illustrations are tabulated to demonstrate the practical usefulness of the method.

This paper is organized in three sections. Using derivative continuities at knots, the consistency relations between the values of spline and its higher order derivatives at knots are determined in first section. In second section the end conditions are derived to complete the definition of spline which completes the required spline solution approximating the solution of the BVP (20). In third section, two examples are considered for the implementation of the method developed. In this paper the method developed not only approximates the solution of BVP, but its higher order derivatives as well. The method developed, provides encouraging results. The possibility of finding the solutions of further higher order BVPs can also be explored in future.

CONCLUSION

This paper is devoted to the evolution, progress, types and spline solutions of differential equations. There is now considerable evidence that in many circumstances a spline function is a more adaptable approximating function than a polynomial involving a comparable number of parameters. Recent trends in computational mathematics, mathematical physics and mechanics are toward the wide use of spline functions to solve such problems. The main advantages of application of spline function are its stability (the local behaviour of a spline at a point does not affect its overall behaviour) and calculation simplicity In solving problems arising in astrophysics, problem of heating of infinite horizontal layer of fluid, eigenvalue problems arising in thermal instability, obstacle, unilateral, moving and free boundary-value problems, problems of the deflection of plates and in a number of other problems of scientific applications, spline functions are not only more accurate but also we have a variety of choices to use quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic or higher splines to solve them.

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