

¹ Fiona QUIRKE, ² Udechukwu OJIAKO, ³ Maxwell CHIPULU

SIMULATING QUEUING SYSTEMS: A TEST OF PARAMETER CHANGE

¹ UNIVERSITY OF SOUTHAMPTON, UNITED KINGDOM

ABSTRACT: This paper examines queuing models, in particular the single-server queuing system. Queuing models assist firms achieve this objective. Queuing models may help firms reduce queues by helping determine what type of system best suits the business. The paper is theory based and uses steady-state equations and simulation to model queuing systems. In particular, the paper examines whether the robustness of steady-state equations and the change of parameters will have a significant effect on queuing models. The paper finds that system complexity does have an impact on the accuracy of the steady-state method results. It is also found that the use of this method is subject to use requirements.

KEYWORDS: queuing models, steady-state equations, simulation, queuing systems

INTRODUCTION

Queuing models can be in the form of both physical and non-physical queues. For example, a queue at a bank is physical, whereas a telephone call to a service centre may be held on a non-physical queue. As firms recognise that customers dislike queuing [1, 2], the desire to maintain relationship with its customers means that firms seek to “manage the trade-off between longer queues, improved service quality and increased costs” [3]. Queuing models assist firms achieve this objective. Queuing models may help firms reduce queues by helping determine what type of system best suits the business.

Queuing models however have their limitations. For example, the classes of algorithms that they handle may be limited in scope and complexity [4]. Queuing models also require independent assumptions to be made which may not be very accurate. For example, the speed the server on average can serve, may differ depending on the day or the customer requirements, yet a decision must be made on how to best describe this distribution overall. As a result, queuing models are often an approximation of a real system. To obtain more accurate characteristics of a real life system, simulation is often recommended [5]. Simulation involves programming specific software to model the queuing system. Even though this is more accurate it can take a long time to implement due to the computer programming that is required because of the detail of information that is required.

Based on this, the objective of this study is to examine a number of existing analytical single-server models and to determine their limitations using simulation software. The single-server models discussed will be tested using a set of theoretical distributions, for both the steady-state equation method and the simulation method, as the simulation method is more accurate; this will be used as a benchmark to show how robust the queuing model is.

QUEUING MODELS AND THEIR ASSUMPTIONS

One of the simplest queuing systems is the standard M/M/1 model which is a single channel model with Poisson arrivals and exponential service times [6]. Poisson probability distribution best describes the distribution pattern and the service time is assumed to follow an exponential distribution.

Assumptions formulas may be developed and used to discover certain aspects of the queue. For these the mean number of arrivals within the time period must be known which is notated by λ and the mean number of services conducted within the period must be known notated by μ . Anderson et al [3], suggest that; Probability that no units are in the system:

$$P_0 = 1 - \frac{\lambda}{\mu} \quad (1)$$

The average number of units in the queue:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad (2)$$

The average number of units in the system:

$$L = L_q + \frac{\lambda}{\mu} \quad (3)$$

The average time a unit spends in the queue:

$$W_q = \frac{L_q}{\lambda} \quad (4)$$

The average time a unit spends in the system:

$$W = W_q + \frac{1}{\mu} \quad (5)$$

The probability that an arriving unit has to wait for service:

$$P_w = \frac{\lambda}{\mu} \quad (6)$$

The probability of n units in the system:

$$p_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad (7)$$

For these to achieve steady-state there must be the constraint that the mean service rate μ is greater than the mean arrival rate λ otherwise the system will grow without a “limit because the service facility does not have sufficient capacity to handle the arriving units” [3] and the steady-state can never be achieved.

THE STUDY

The experiment uses two quantitative methods to model queues. These methods are the steady-state equation method which is conducted on Microsoft Excel and the simulation method which is preformed using a software package called Simul8.

Simul8 allows randomly generated trials to be performed based on the numbers imputed into the model. It also enables users alter the service and arrival probability functions, change the queue discipline and set a capacity on the queue, which is required to produce results that represent each independent queue model. To test the parameters of each queuing model, a set of scenarios will be conducted for each queuing model. The steady-state equations require the service and arrival rate over a certain time period which is one hour for this experiment, whereas the Simul8 software requires the time between arrivals and the time it takes to serve one unit. For example, if $\mu=25$ and $\lambda= 20$ for the steady-state method, then the numbers required for a compatible simulation would be $\mu=2.4$ and $\lambda= 3$ which is 60 divide the steady-state equations inputs. We note that as long the numbers imputed represent the same distribution for each method then the comparison is valid. The input numbers will be consistent, so that comparison can occur between the simulation and steady-state approach for each queuing system.

Using the Simul8 package, a model of the single-server queue was made; which has a work entry point, a storage bin representing the queue, a work centre followed by exit point, all connected by routing arrows. Once the basic model was formed and the correct probabilities and numbers imputed, the duration time was set. Although Simul8 is automatically set to the working week, this experiment required a non-terminating model to be simulated (to be comparable to the steady-state method). For this reason, the duration was set to 480 minutes which is one continual working day. Simul8 also incorporates the transient period with in its results, which must be excluded. To allow for valid comparison, the warm-up period was set. To determine the warm-up period, the results were plotted against time. To overcome the problem of auto-correlation, multiple runs were performed, thus ensuring that the random numbers generated for each run do not correlate and are independent of each

other. This is because if autocorrelation occurs, it will impact on the reliability of the method. Another reason why multiple runs are required is to give an appropriate average for the model, so that extreme results don't affect the overall average. The number of trials performed was determined by the confidence level set, which for this experiment was 95%. Simul8 can automatically run the correct number of trials to achieve this confidence level, therefore ensuring the reliability of the results. The confidence level of 95% was based on 4 different results; average queue size, average use, average time in system and average queuing time.

To analyze the results and to justify the aim of the experiment, which is based on the comparison of two quantitative methods to model queues (using Microsoft Excel and Simul8), a direct comparison was performed using the simulation results as a benchmark. This indicated how the results varied. The difference between the average results for each method was taken and averaged. t-tests were also conducted using Microsoft Excel on the two different methods results, to show whether their difference was significant. The t-test was be “two-tailed” with a null hypothesis that ‘both methods are equal in predicting queue characteristics’ and the alternative hypothesis ‘that the steady-state method is inaccurate and differs.’

RESULTS AND ANALYSIS

The M/M/1 SIRO model was not tested because it can't be easily modelled via Simul8. As it is not common to choose people randomly from a queue, this requirement was excluded from the experiment. Another model that has been discussed, but not experimented on is the G/M/1 model, this is because its formulas directly predict the aspects of the queue and not the average, and therefore comparison couldn't occur.

M/M/1 Model analysis

Examining the results from M/M/1 show that all 4 queue characteristic results differ between the simulation and steady-state method (shown in Figure 1). The t-tests designed to test the significance between the two methods results, examines whether the null hypothesis is still valid or if the alternative is now accepted, as stated in the methodology.

Table 1. M/M/1 t-test

| | L_q | L | W | W_q |
|------------------|-------------|-------------|-------------|-------------|
| T-statistic | 4.311968995 | 4.196175071 | 3.56127571 | 4.733482978 |
| P-Value | 0.001956039 | 0.002319796 | 0.006107069 | 0.00106866 |
| Critical T-Value | 2.262157158 | 2.262157158 | 2.262157158 | 2.262157158 |

From Table 1, it can be seen that the difference between the two methods is significant for all the characteristic results. The reason for this is because

the t-statistic for all 4 is significantly greater than the critical value and the p-value is lower than the significance level of 0.05. Examining the summarized t-test table the W_q results are seen to have the most significant difference because its p-value is the lowest at 0.001(3dp) and it has the greatest difference between the critical t value and the t-statistic of 2.47 (2dp).

| M/M/1 | | Lq | | L | |
|-----------|-------|------------------|---------------------|-------------|------------------|
| λ | μ | Simulation | Steady-state method | Difference | Simulation |
| 20 | 25 | [1.68,1.77,1.86] | 3.2 | 1.43 | [2.45,2.54,2.64] |
| 20 | 30 | [0.67,0.71,0.74] | 1.333333333 | 0.623333333 | [1.29,1.33,1.37] |
| 20 | 35 | [0.34,0.36,0.38] | 0.761904762 | 0.40190476 | [0.84,0.87,0.9] |
| 15 | 20 | [1.09,1.15,1.2] | 2.25 | 1.1 | [1.83,1.89,1.95] |
| 15 | 25 | [0.4,0.42,0.44] | 0.9 | 0.48 | [0.98,1.01,1.04] |
| 15 | 30 | [0.19,0.2,0.21] | 0.5 | 0.3 | [0.67,0.68,0.7] |
| 10 | 15 | [0.56,0.59,0.62] | 1.333333333 | 0.743333333 | [1.24,1.28,1.32] |
| 10 | 20 | [0.18,0.19,0.2] | 0.5 | 0.31 | [0.71,0.72,0.74] |
| 10 | 25 | [0.08,0.08,0.09] | 0.266666667 | 0.186666667 | [0.51,0.52,0.53] |
| 10 | 30 | [0.04,0.04,0.04] | 0.166666667 | 0.126666667 | [0.41,0.41,0.41] |
| | | Average | 0.57019048 | | 0.575 |

| W | | Wq | |
|------------------|---------------------|------------|------------------|
| Simulation | Steady-state method | Difference | Simulation |
| [8.21,8.47,8.73] | 12 | 3.53 | [5.29,5.54,5.8] |
| [4.77,4.89,5] | 6 | 1.11 | [2.24,2.34,2.44] |
| [3.45,3.51,3.57] | 4 | 0.49 | [1.21,1.26,1.31] |
| [8.03,8.27,8.5] | 12 | 3.73 | [4.51,4.73,4.95] |
| [4.66,4.76,4.85] | 6 | 1.24 | [1.72,1.8,1.88] |
| [3.41,3.46,3.51] | 4 | 0.54 | [0.87,0.91,0.95] |
| [7.79,7.99,8.18] | 12 | 4.01 | [3.32,3.49,3.66] |
| [4.6,4.68,4.75] | 6 | 1.32 | [1.09,1.14,1.2] |
| [3.4,3.44,3.47] | 4 | 0.56 | [0.48,0.5,0.53] |
| [2.78,2.8,2.83] | 3 | 0.2 | [0.26,0.27,0.28] |
| | | Average | 1.673 |

Figure 1. M/M/1 Result

Examining the L_q and W_q results shows that changing the service rate by 5 can drastically change the results; take distribution $\lambda=20$ and $\mu=25$ compared to $\lambda=15$ and $\mu=25$. Between these two scenarios a decrease of over 60% has occurred, showing that the parameters have a significant effect on the model and its results. The simulation results for L_q also went from the average of 1.77 to 0.42 due to the change in the arrival rate between this periods.

M/M/1 Finite Model analysis

As with the standard M/M/1 model, all of the characteristics significantly differed, with the p-values being under the significance level of 0.05. However, with the M/M/1 Finite model (Figure 2), the difference was more significant as seen in the t-test which shows the highest p-value to be 0.0000134308 (Table 2).

The difference between its critical value and t-statistic is 6.25 (2dp) which is also greater than any other models. It contains the biggest difference between the t-statistic and the critical value of 14.32 (2dp) for L which is the most significant difference of the model with a p-value of 0.0000000471623. The steady-state method is severely flawed and inaccurate because on

average it has overestimated the units in the queue by 2.71(2dp) and their wait by 5.83(2dp).

| M/M/1 finite | | Lq | | L | |
|--------------|---|---------|------------------|---------------------|-------------|
| λ | N | μ | Simulation | Steady-state method | Difference |
| 20 | 6 | 25 | [1.24,1.31,1.37] | 3.752693966 | 2.442693966 |
| 20 | 6 | 30 | [0.61,0.65,0.68] | 3.503591954 | 2.853591954 |
| 15 | 6 | 20 | [0.96,1.01,1.06] | 3.669646658 | 2.659646658 |
| 15 | 6 | 25 | [0.39,0.41,0.43] | 3.337590464 | 2.927590464 |
| 10 | 6 | 15 | [0.56,0.59,0.62] | 3.503591954 | 2.913591954 |
| 10 | 6 | 20 | [0.18,0.19,0.2] | 3.005747126 | 2.815747126 |
| 20 | 7 | 25 | [1.35,1.42,1.5] | 4.750510621 | 3.330510621 |
| 15 | 7 | 20 | [1.01,1.06,1.12] | 4.667231502 | 3.607231502 |
| 15 | 5 | 20 | [0.9,0.95,1] | 2.678021087 | 1.728021087 |
| 10 | 5 | 20 | [0.18,0.19,0.2] | 2.02189781 | 1.83189781 |
| | | Average | 2.711052314 | | 3.028902852 |

| W | | Wq | |
|------------------|---------------------|-------------|------------------|
| Simulation | Steady-state method | Difference | Simulation |
| [7.27,7.5,7.73] | 11.41726205 | 3.91726205 | [4.09,4.3,4.5] |
| [4.87,4.98,5.09] | 9.017266187 | 4.037266187 | [2.09,2.19,2.28] |
| [7.88,8.11,8.34] | 14.0230179 | 5.9130179 | [4.09,4.29,4.5] |
| [4.89,4.99,5.08] | 10.42302526 | 5.43302526 | [1.7,1.78,1.86] |
| [8.09,8.29,8.5] | 18.03453237 | 9.74453237 | [3.33,3.51,3.68] |
| [4.85,4.93,5] | 12.03454894 | 7.10454894 | [1.09,1.15,1.2] |
| [7.59,7.83,8.07] | 13.8038135 | 5.9738135 | [4.41,4.63,4.85] |
| [8.03,8.26,8.49] | 17.00508475 | 8.74508475 | [4.24,4.46,4.67] |
| [7.66,7.88,8.1] | 11.07334963 | 3.19334963 | [3.86,4.06,4.26] |
| [4.84,4.92,5] | 9.110294118 | 4.19029412 | [1.08,1.14,1.19] |
| | | Average | 5.82521947 |

Figure 2. M/M/1 Finite Result

Table 2. M/M/1 Finite t-test

| T- | L_q | L | W | W_q |
|------------------|-------------|-------------|-------------|-------------|
| statistic | 11.62901644 | 16.57898294 | 8.512696385 | 9.703550601 |
| P-Value | 1.00568E-06 | 4.71623E-08 | 1.34308E-05 | 4.59414E-06 |
| Critical T-Value | 2.262157158 | 2.262157158 | 2.262157158 | 2.262157158 |

On examining the parameters, it is observed that $\lambda=15$, $\mu=20$ and the changing the N value to 5,6 and 7 a rounded increase of 1 between each change occurred using the steady-state method for the L and L_q characteristics, as shown in figure 4.4. It also resulted in a change of 0.06 between N=5 and N=6 and 0.05 between N=6 and N=7 using the simulation method. For W and W_q the same applies with a difference of 3 occurring with the steady-state method. Overall this model is shown to be the least robust concerning the steady-state method.

M/M/1 LIFO Model analysis

M/M/1 LIFO resulted in the same steady-state equation results as the standard M/M/1 model and the simulation results were very similar to the M/M/1 models. Similar t-test results were also observed. The average difference for L_q was show to be 0.58(2dp) units and the average difference for W_q was 2.3 minutes (see Table 3). The simulation results took into account slight changes, even if this was only 0.02 units, which is the difference between the two models L_q results where $\lambda=20$, $\mu=25$.

Table 3. M/M/1 LIFO t-test

| | L_q | L | W | W_q |
|------------------|-------------|-------------|-------------|-------------|
| T-statistic | 4.252385054 | 4.153794538 | 3.527840075 | 4.623601975 |
| P-Value | 0.002134956 | 0.0024704 | 0.006435727 | 0.001247953 |
| Critical T-Value | 2.262157158 | 2.262157158 | 2.262157158 | 2.262157158 |

M/G/1 Model analysis

From these models results, it is observed that the average difference for the characteristic was slightly lower than that of the standard M/M/1 model. The average difference for the characteristic was also less significant. For example, for W, a p-value of 0.315(3dp) is observed (Table 4). This is above the 0.05 significance level. At the same time, the t- statistic value 1.065(3dp), is below the two tailed critical value. The most significant difference came from the average waiting time in the queue which had a t- statistic value of 0.725(3dp) which is greater than the critical value. The p-value of 0.015(3dp), even though being significant is small compared to the models previously discussed, thus showing M/G/1 to be a more robust model.

Table 4. M/G/1t-test

| | L_q | L | W | W_q |
|------------------|-------------|-------------|-------------|-------------|
| T-statistic | 2.959748927 | 2.437143552 | 1.065238156 | 2.987433563 |
| P-Value | 0.015966028 | 0.037540114 | 0.314510559 | 0.015264341 |
| Critical T-Value | 2.262157158 | 2.262157158 | 2.262157158 | 2.262157158 |

As with the other models, changing the parameters did impact the results. By changing the standard deviation, it is observed in Figure 3, that the average for all the characteristics decreased.

Within the results it can be seen that the steady-state method has underestimated some characteristics compared to the simulation approach. For W when $\lambda=20$, $\mu=25$, $\sigma=0.025$ it is 1.945 lower than the simulation average which is a more accurate approach. This underestimation could result in the user underestimating the resources required for the task, resulting in negative side effects.

M/D/1 Model analysis

All of M/D/1 characteristic results differ significantly, as shown in the t-test highlights (Table 5), which have p-values under 0.05 and a t-statistic greater than the critical value.

Table 5, M/D/1 t-test

| | L_q | L | W | W_q |
|------------------|------------|-------------|-------------|-------------|
| T-statistic | 4.0266627 | 3.475546778 | 3.090027026 | 4.486895995 |
| P-Value | 0.0029879 | 0.006987412 | 0.012928908 | 0.001517281 |
| Critical T-Value | 2.26215715 | 2.262157158 | 2.262157158 | 2.262157158 |

The largest significance was observed in W_q which had a t-statistic of 2.225 over the critical value and a p-value of 0.0015(4dp). This shows that there is a significant difference between the two methods and the steady-state method is not that robust at efficiently predicting the characteristics of the queue.

M/E_k/1 Model analysis

The results of the M/E_k/1 Model analysis (Figure 4) was similar to the previous models. The exception was however in the M/G/1. The results of the M/E_k/1 Model analysis shows that the average number of units in the system, L, was the most significant with a p-value of 0.0022(5dp) and a t- statistic 3.677(3dp) greater than the critical value.

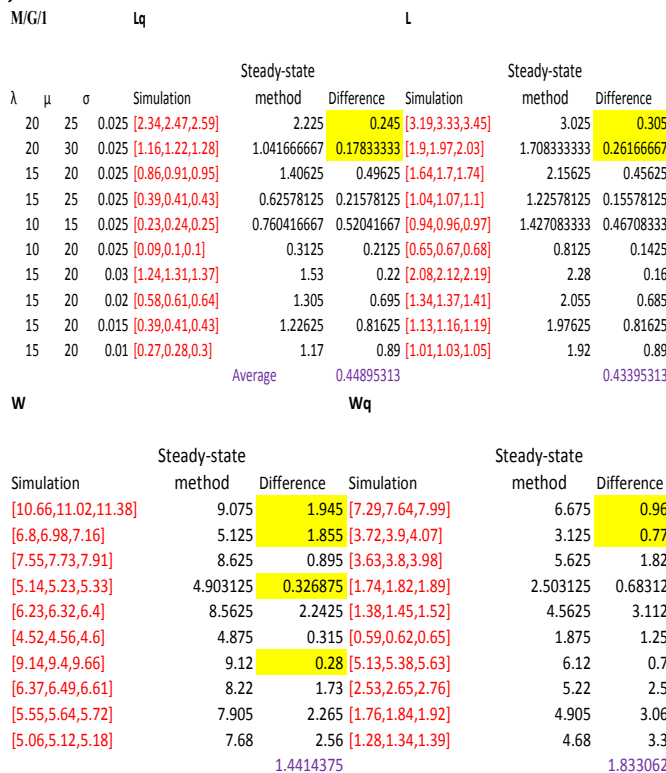


Figure 3. M/G/1 Result

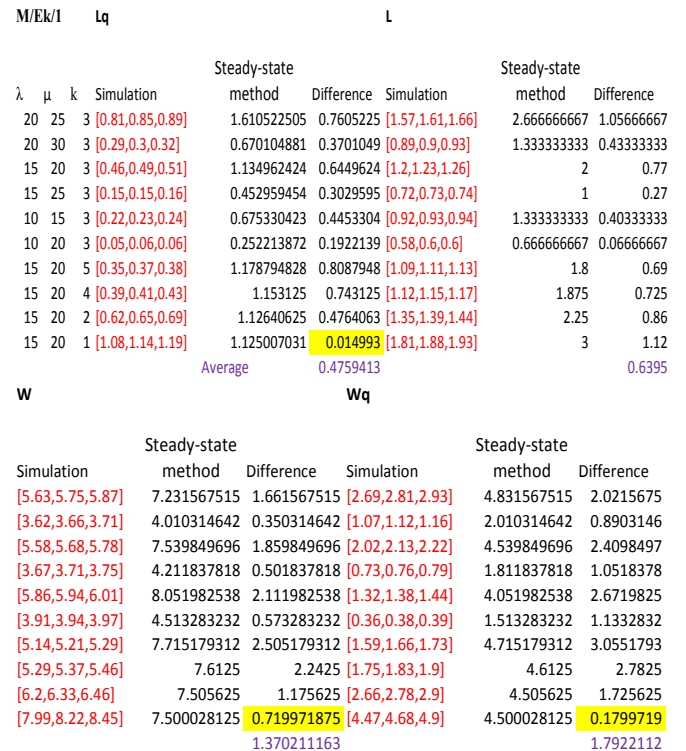


Figure 4. M/E_k/1 Result

Table 6. $M/E_k/1$ -test

| | L_q | L | W | W_q |
|------------------|-------------|-------------|-------------|-------------|
| T-statistic | 5.54287921 | 5.939409466 | 3.763802306 | 5.416771379 |
| P-Value | 0.000359712 | 0.000218172 | 0.004458812 | 0.00042362 |
| Critical T-Value | 2.262157158 | 2.262157158 | 2.262157158 | 2.262157158 |

By changing the parameters it can be seen that via the simulation approach the more phases (higher k value), the less time or units are in the system or queue. Between scenarios $\mu=20$, $\lambda=15$ and $K=5$ to $\mu=20$, $\lambda=15$ and $K=1$, the average units in the queue has decreased by 0.77 units. These findings show that the steady-state method is insufficient in estimating the systems characteristic, especially as it doesn't reflect the fact the more phases the system has the lower the results.

ADD-ON TEST, RESULTS AND ANALYSIS

Simul8 was utilized to test a variety of probability functions which have not be previously used. The objective being to test the robustness of the original model in relation to changing the probability functions. A total of 20 different scenarios which had similar input values were considered.

An examination of the Add-on results showed that using simulation allowed the standard $M/M/1$ model to be very flexible due to the ease of its modification. However, using the binomial for the arrival rate and exponential for the service rate, resulted in significantly larger results compared to the other exponentially distributed service rate models which had an L_q average that varied between 0.05 and 2.98 (whereas for this scenario it was 15.4. The fact that this model's results differ significantly from the others shows that changing the probability distribution can have a serious impact on the results and therefore managers must ensure they choose the correct type of probability function to use when modeling a system.

When the service rate was changed, a similar result occurred with the combination of Poisson arrival rate and gamma service rate. The average delay in the system was 87.13 minutes, whereas the other Poisson arrival models varied from 0.84 minutes to 5.95, excluding the model with negative binomial which had an average of 23.55. The model with a negative binomial service rate also shows a significant difference, but this is small compared to the gamma service rate model. Both imply that the steady-state is not being me.

The $M/M/1$ model is shown to be very adaptable using simulation primarily because of its ease of use, especially when inserting new distributions into the standard model. This is contradictory to the steady-state equation method. Here, for every variation of the original model new formulas need to be determined.

CONCLUSIONS

The study shows that the steady-state equation is not robust as only the $M/G/1$ model contained an insignificant difference between the two methods. Even so, only one characteristic was insignificant out of the 4 different results that is average queue size, average use, average time in system and average queuing time. The experiment also showed that changing the parameters does have a significant impact on the results, especially if the probability distribution is changed, as seen in the Add-on experiment. Overall the experiment showed the steady-state equation method to be very limited, as even the standard $M/M/1$ model was inaccurate compared to the simulation method. Although this is the case, the use of the steady-state equation method can still be justified. This is because organizations will need to make a compromise on whether the increase cost and time that is associated with the use of simulation software is worth the improved accuracy of the data.

Overall the findings are consistent with the theory that the more complex the queuing model, the less accurate the results are, however, it was surprising to discover that even the standard $M/M/1$ model had a significant difference. It showed the two most complex models to be the least accurate and as previously stated has recommended the use of simulation instead of the steady-state equations.

REFERENCES

- [1.] KELLY, F (1975). Networks of Queues with Customers of Different Types. *Journal of Applied Probability*, Vol. 12, No. 3, pp. 542-554.
- [2.] SHEU, C and MCHANEY, R (2003). Service process design flexibility and customer waiting time. *International Journal of Operations & Production Management*. Vol. 23, No. 8, pp. 901-917.
- [3.] ANDERSON, D., SWEENEY, D., WILLIAMS, T., and WISNIEWSKI, M (2009). *An Introduction to Management Science*. 12th ed. South-Western: China.
- [4.] DEFRAEYE, M and VAN NIEUWENHUYSE, I (2011). Setting staffing levels in an emergency department: opportunities and limitations of stationary queuing models. *Review of Business and Economics*, Vol. 56, No. 1, pp. 73-100.
- [5.] BENNETT, R (1998). Queues, customer characteristics and policies for managing waiting-lines in supermarkets. *International Journal of Retail & Distribution Management*, Vol. 26, No. 2, pp. 78 – 87.
- [6.] MILLER, D (1981). Computation of Steady-State Probabilities for $M/M/1$ Priority Queues. *Operations Research*, Vol. 29, No. 5, pp. 945-958.

